A characterization of the Einstein tensor

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Abstract

In order to motivate the field equation of General Relativity, there exists a theorem, already used by Einstein in his foundational works, that computes all the symmetric and divergence free 2-covariant "natural" tensors (i.e., invariant by the action of local diffeomorphisms) whose coefficients are constructed from the coefficients of the metric g, its first and second derivatives and depend linearly on these second derivatives. Such tensors are just the linear combinations:

$$\lambda \left(R_2 - \frac{1}{2} r g \right) + \Lambda\, g$$

where R_2 and r stand for the Ricci tensor and the scalar curvature of g respectively, and $\lambda, \Lambda \in \mathbb{R}$.

In the 1970's, it was showed that, in dimension 4, the assumptions of symmetry and linearity in the second derivatives of the metric are superfluous (Lovelock theorem).

Our objective is to improve these results, eliminating the hypothesis of symmetry and the restriction on the order of the derivatives of the metric in arbitrary dimension.

Concretely, using the theory of natural bundles, we characterize the Einstein tensor $G_2(g) := R_2 - \frac{1}{2} r g$ as the only (up to a constant factor) 2-covariant natural tensor associated to a semi-riemannian metric which is divergence free and has weight 0 (that is, satisfies $G_2(\lambda^2 g) = G_2(g)$ for all $\lambda \in \mathbb{R}^+$).