3-quasi-Sasakian manifolds

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(joint work with B. Cappelletti Montano and G. Dileo)

Quasi-Sasakian manifolds were introduced by D. E. Blair in [3] in the attempt to unify Sasakian and cosymplectic geometry. Blair defined a quasi-Sasakian structure on a $(2n+1)$-dimensional manifold M as a normal almost contact metric structure (ϕ, ξ, η, g) whose fundamental 2-form Φ is closed. The analogue of quasi-Sasakian manifolds in the setting of 3-structures is given by the class of 3-quasi-Sasakian manifolds which include as special cases the 3-cosymplectic and the 3-Sasakian manifolds. Recently these manifolds have been the subject of a growing interest (see the survey [5] and references therein) in view of the finding of significant applications to physics, in particular to supergravity and M-theory (see e.g. [1, 2, 8]).

In this note we briefly present the main results of our systematic study [6] of 3-quasi-Sasakian manifolds. A 3-*quasi-Sasakian manifold* is an almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that each fundamental 2-form Φ_α is closed and each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal. The first step in our work is to prove that the distribution spanned by the three characteristic vector fields ξ_1, ξ_2, ξ_3 is involutive and defines a 3-dimensional Riemannian and totally geodesic foliation $\mathcal{V}$ of M. Taking into account the geometry of $\mathcal{V}$ we show that 3-quasi-Sasakian manifolds divide into two classes: those manifolds for which the foliation $\mathcal{V}$ has the local structure of an abelian Lie group, and those for which $\mathcal{V}$ has the local structure of the Lie group $SO(3)$ (or $SU(2)$). For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. We prove that these ranks coincide, allowing us to classify 3-quasi-Sasakian manifolds according to their well-defined rank, which is of the form $4p+1$ in the abelian case and $4p+3$ in the non-abelian one. Note that 3-cosymplectic manifolds (rank 1) and 3-Sasakian manifolds (rank $4n+3 = \dim(M)$) are two representatives of each of the above classes. Nevertheless we show examples of 3-quasi-Sasakian manifolds which are neither 3-cosymplectic nor 3-Sasakian. Furthermore, we prove a splitting theorem for any 3-quasi-Sasakian manifold M assuming the integrability of one of the almost product structures. We prove that if M belongs to the class of 3-quasi-Sasakian manifolds with $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$, then M is locally the product of a 3-Sasakian and a hyper-Kählerian manifold, whereas if M belongs to the class of 3-quasi-Sasakian manifolds with $[\xi_\alpha, \xi_\beta] = 0$, then M is 3-cosymplectic. Finally, we find an application of the integrability of the vertical distribution showing that $\mathcal{V}$ is a minimum of the corrected energy in the sense of Chacón-Naveira ([7]) and Blair-Turgut Vanli ([4]).

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